Analysis of trapped modes in beam pipes of accelerating cavity modules

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Abstract

This contribution investigates an important aspect of the design of particle accelerators, i.e., the behaviour of the geometric shunt impedance, R/Q, as a function of the beam pipe length for eigenmodes formed within the beam pipes connecting the cavities in a module. The study employs both analytical and numerical approaches to characterise these dependencies, providing an understanding of how R/Q varies with the geometric parameters. Analytical formulas are derived and validated by comparison with numerical solutions. The results highlight optimal performance - minimised impedance - for certain beam pipe length-to-radius ratios.

1 Introduction

Accelerating cavities are critical components of particle accelerators designed to increase the velocity and energy of charged particles. These cavities are typically arranged in modules, interconnected by cylindrical waveguides known as beam pipes. Beam pipes can support the formation of longitudinal and transverse modes with high impedance, which can compromise beam stability if not sufficiently damped.

The impedance of these modes is determined through eigenmode analysis as the product of the geometric shunt impedance (R/Q) and the quality factor (Q). Couplers can be used to reduce Q, while R/Q is purely a function of the component's geometry and can be minimised by optimising its geometric parameters.

This study examines how the dimensions of the beam pipes influence R/Q for a cylindrical waveguide as a precursor to analysing beam pipes coupled to cavities. The focus is on transverse magnetic (TM) monopole modes, with particular attention to the TM_{011} mode, which is illustrated in Fig. 1.

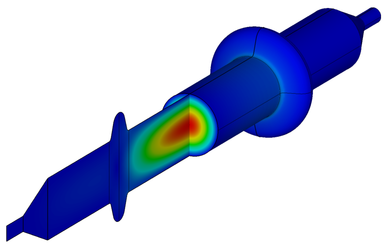

Figure 1: Electric field distribution of a TM_{011} mode trapped in the beam pipe connecting two single-cell cavities.

2 Analytical formalism

As a precursor to analysing beam pipes of any desired shape coupled to cavities, a general solution is derived to describe the relationship between the R/Q of a cavity eigenmode and the length-to-radius ratio (L/R) of a cylindrical waveguide. Figure 2 illustrates the cylindrical waveguide mode corresponding to the mode shown in Fig. 1.

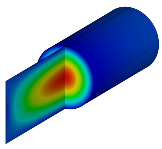

Figure 2: Electric field distribution of a TM_{011} mode of a cylindrical waveguide.

Using cylindrical coordinates, the R/Q for an eigenmode is defined as [1]

$$\frac{R}{Q} := \frac{V_{\parallel}(\rho = 0)^2}{\omega U}, \tag{1}$$

where R is the shunt impedance, Q is the quality factor, $V_{\parallel}(\rho = 0)$ is the central axis voltage, ω is the angular frequency and U is the energy of the mode. The accelerating voltage and energy are calculated using

$$V_{\parallel}(\rho = 0) = \int_0^L E_z(\rho = 0, \phi) e^{j\omega t} \mathrm{d}z, \tag{2}$$

$$U = \frac{1}{2}\epsilon_0 \int_V \mathbf{E} \cdot \mathbf{E} dV = \frac{1}{2}\mu_0 \int_V \mathbf{H} \cdot \mathbf{H} \mathrm{d}V, \tag{3}$$

respectively. $E_z(\rho, \phi)$ represents the z-component of the phasor electric field vector $\mathbf{E}$, while $\mathbf{H}$ denotes the phasor magnetic field vector, L denotes total length, and k_z is the z propagation constant. The expression for the longitudinal component of the electric field for transverse magnetic (TM) modes in a cylindrical waveguide is derived as shown in [2, p. 128]. Only relevant quantities are repeated here.

$$E_z = A\cos(m\phi)\cos(k_z z) J_m(k_{\mathrm{c},mn}\rho) \tag{4a}$$

$$H_\rho = jA\frac{\omega\epsilon m}{k_{\mathrm{c},mn}^2\rho}\sin(m\phi)\cos(k_z z)J_m(k_{\mathrm{c},mn}\rho) \tag{4b}$$

$$H_\phi = jA\frac{\omega\epsilon}{k_{\mathrm{c},mn}}\cos(m\phi)\cos(k_z z)J_m'(k_{\mathrm{c},mn}\rho) \tag{4c}$$

where ϵ and μ are the permittivity and permeability of the medium, A is an arbitrary amplitude constant, m, n, and p are the number of field variations in the azimuthal ϕ, radial ρ and longitudinal z axes, respectively, $k_{\mathrm{c},mn}$ the cutoff frequency, and J_m a Bessel function of the m^{th} kind.

Substituting (4a) in (2) we get,

$$V_{\|}(\rho = 0) = E_0\cos(m\phi)\int_0^L \cos(k_z z)J_m(0)e^{j\omega t}\mathrm{d}z. \tag{5}$$

For TM modes, $H_z = 0$. Therefore,

$$U = \frac{1}{2}\mu_0\int_V (H_\rho H_\rho^* + H_\phi H_\phi^*)\mathrm{d}V. \tag{6}$$

Substituting (4c) and (4b) in (6), we get

$$U = \frac{1}{2}\left(\frac{\omega\mu_0\epsilon E_0}{k_{\mathrm{c},mn}^2}\right)^2\int_V \left[\frac{m^2}{\rho^2}\sin^2(m\phi)\cos^2(k_z z)J_m^2(k_{\mathrm{c},mn}\rho) + k_{\mathrm{c},mn}^2\cos^2(m\phi)\cos^2(k_z z)J_m'^2(k_{\mathrm{c},mn}\rho)\right]\rho\,\mathrm{d}\phi\,\mathrm{d}\rho\mathrm{d}z. \tag{7}$$

Using the Bessel's integral identity in (7),

$$\int_0^R \frac{m^2}{\rho^2}J_m^2(k_{\mathrm{c},mn}\rho) + k_{\mathrm{c},mn}^2 J_m'^2(k_{\mathrm{c},mn}\rho)\rho\mathrm{d}\rho = \frac{p_{mn}^2}{2}J_m'^2(p_{mn}), \tag{8}$$

where $p_{mn} := k_{\mathrm{c},mn}R$ for radius R, results in,

$$U = \frac{1}{2^i}\pi\epsilon L E_0^2\frac{\kappa^2R^2}{p_{mn}^2}J_m'^2(p_{mn}). \tag{9}$$

where

$$i = \begin{cases} 1 & \text{if } m = 0, p = 0 \\ 2 & \text{if } m = 0, p > 0 \text{ or } p = 0, m > 0 \\ 3 & \text{if } m, p > 0 \end{cases} \tag{10}$$

From (2) and (9), R/Q is calculated as,

$$\frac{R}{Q} = \frac{2^{i-2}p_{mn}^2\eta L\cos(m\phi)J_m(0)(T_L + T_R)^2}{\pi\kappa^3 R J_m'^2(p_{mn})}, \tag{11}$$

where $\eta = \mu/\epsilon$ is the characteristic impedance of the propagation medium and $\kappa := R\sqrt{k_{\mathrm{c},mn}^2 + k_z^2}$.

For TM_{0np} modes, m = 0,

$$\frac{R}{Q} = \frac{2^{i-2}p_{0n}^2\eta L(T_L + T_R)^2}{\pi\kappa^3 R J_0'^2(p_{0n})}, \tag{12}$$

where

$$T_{L/R} = \frac{\sin(\frac{\psi_{L/R}}{2})}{\frac{\psi_{L/R}}{2}}; \quad \psi_{L/R} = L\left(\frac{\omega}{c} \pm k_z\right), \tag{13}$$

From (12), it is observed that R/Q is minimum (=0) only when $T_L + T_R = 0$. Substituting the definitions for T_L and T_R from (13) and using some trigonometric identities, we arrive at

$$\sin\left(\frac{\kappa L}{R}\right)\cos\left(\frac{p\pi}{2}\right) + \sin\left(\frac{p\pi}{2}\right)\cos\left(\frac{\kappa L}{R}\right) = 0. \tag{14}$$

If p is odd,

$$\frac{L}{R} = \frac{\pi}{p_{mn}}\sqrt{n^2 - p^2} : n \in \{1, 3, 5, ...\}. \tag{15}$$

If p is even,

$$\frac{L}{R} = \frac{\pi}{p_{mn}}\sqrt{(2n)^2 - p^2} : n \in \{1, 2, 3, 4, ...\}. \tag{16}$$

The result shows that R/Q is minimised (=0) for discrete L/R ratios. The optimal L/R ratios for the beam pipes can be calculated for various TM_{0np} modes using equations (15) and (16).

3 Validation

To validate the results from the analytical formalism, a cylindrical cavity is constructed in CST Studio Suite® [3] and simulated for different L/R ratios. The analytical and numerical results, shown in Fig. 3, demonstrate good agreement, thereby validating the derived formula.

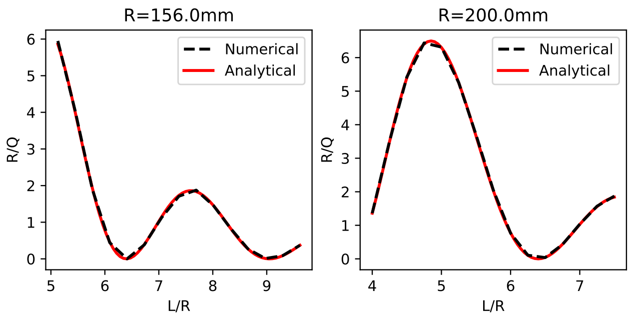

Figure 3: Comparison of analytical and numerical solutions for relationship between L/R vs R/Q for R = 156 mm and 200 mm for the TM_{011} mode.

4 Conclusion and outlook

This study shows that for specific beam pipe length-to-radius L/R ratios, the geometric shunt impedance R/Q of a cylindrical waveguide is minimised. The result could be leveraged to carefully define the cavity module length to reduce the adverse effect of the beam pipe modes trapped between cavities in a module. The next step is comparing the analytic solution for the cylindrical waveguide to numerical solutions for a beam pipe between cavities.

References

[1] H. Padamsee, RF Superconductivity for Accelerators, John Wiley & Sons, Inc., 1998.

[2] D. Pozar, Microwave Engineering, Fourth Edition, John Wiley & Sons, Inc., 2012.

[3] CST Studio Suite®, `https://www.3ds.com/products-services/simulia/products/cst-studio-suite/electromagnetic-systems`.